

# Module 4 Single-phase AC Circuits

# Lesson 17

## Resonance in Series and Parallel Circuits

In the last lesson, the following points were described:

1. How to compute the total impedance in parallel and series-parallel circuits?
2. How to solve for the current(s) in parallel and series-parallel circuits, fed from single phase ac supply, and then draw complete phasor diagram?
3. How to find the power consumed in the circuits and also the different components, and the power factor (lag/lead)?

In this lesson, the phenomenon of the resonance in series and parallel circuits, fed from single phase variable frequency supply, is presented. Firstly, the conditions necessary for resonance in the above circuits are derived. Then, the terms, such as bandwidth and half power frequency, are described in detail. Some examples of the resonance conditions in series and parallel circuits are presented in detail, along with the respective phasor diagrams.

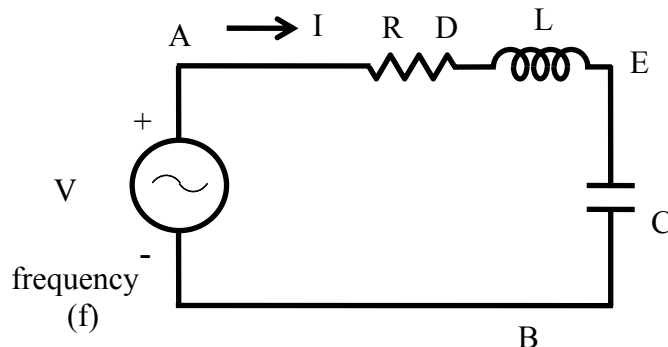
**Keywords:** Resonance, bandwidth, half power frequency, series and parallel circuits,

After going through this lesson, the students will be able to answer the following questions;

1. How to derive the conditions for resonance in the series and parallel circuits, fed from a single phase variable frequency supply?
2. How to compute the bandwidth and half power frequency, including power and power factor under resonance condition, of the above circuits?
3. How to draw the complete phasor diagram under the resonance condition of the above circuits, showing the currents and voltage drops in the different components?

## Resonance in Series and Parallel Circuits

### Series circuit



**Fig. 17.1 (a) Circuit diagram.**

The circuit, with resistance  $R$ , inductance  $L$ , and a capacitor,  $C$  in series (Fig. 17.1a) is connected to a single phase variable frequency ( $f$ ) supply.

The total impedance of the circuit is

$$Z \angle \phi = R + j \left( \omega L - \frac{1}{\omega C} \right)$$

where,

$$Z = \sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2} ; \quad \phi = \tan^{-1} \frac{(\omega L - 1/\omega C)}{R} ; \quad \omega = 2\pi f$$

The current is

$$I \angle -\phi = \frac{V \angle 0^\circ}{Z \angle \phi} = (V/Z) \angle -\phi$$

$$\text{where } I = \frac{V}{\left[ R^2 + (\omega L - (1/\omega C))^2 \right]^{1/2}}$$

The current in the circuit is maximum, if  $\omega L = \frac{1}{\omega C}$  .

The frequency under the above condition is

$$f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

This condition under the magnitude of the current is maximum, or the magnitude of the impedance is minimum, is called resonance. The frequency under this condition with the constant values of inductance L, and capacitance C, is called resonant frequency. If the capacitance is variable, and the frequency,  $f$  is kept constant, the value of the capacitance needed to produce this condition is

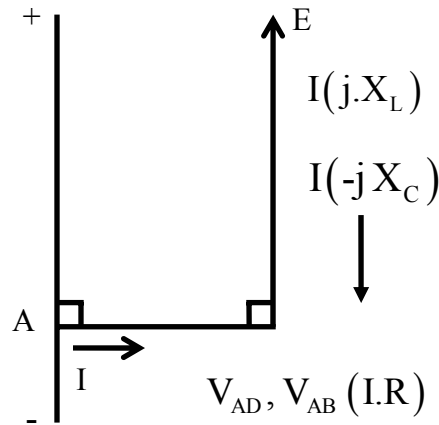
$$C = \frac{1}{\omega^2 L} = \frac{1}{(2\pi f)^2 L}$$

The magnitude of the impedance under the above condition is  $|Z| = R$ , with the reactance  $X = 0$ , as the inductive reactance  $X_L = \omega L$  is equal to capacitive reactance  $X_C = 1/\omega C$ . The phase angle is  $\phi = 0^\circ$ , and the power factor is unity ( $\cos \phi = 1$ ), which means that the current is in phase with the input (supply) voltage.. So, the magnitude of the current ( $|V/R|$ ) in the circuit is only limited by resistance, R. The phasor diagram is shown in Fig. 17.1b.

The magnitude of the voltage drop in the inductance L/capacitance C (both are equal, as the reactance are equal) is  $I \cdot \omega_o L = I \cdot (1/\omega_o C)$ .

The magnification of the voltage drop as a ratio of the input (supply) voltage is

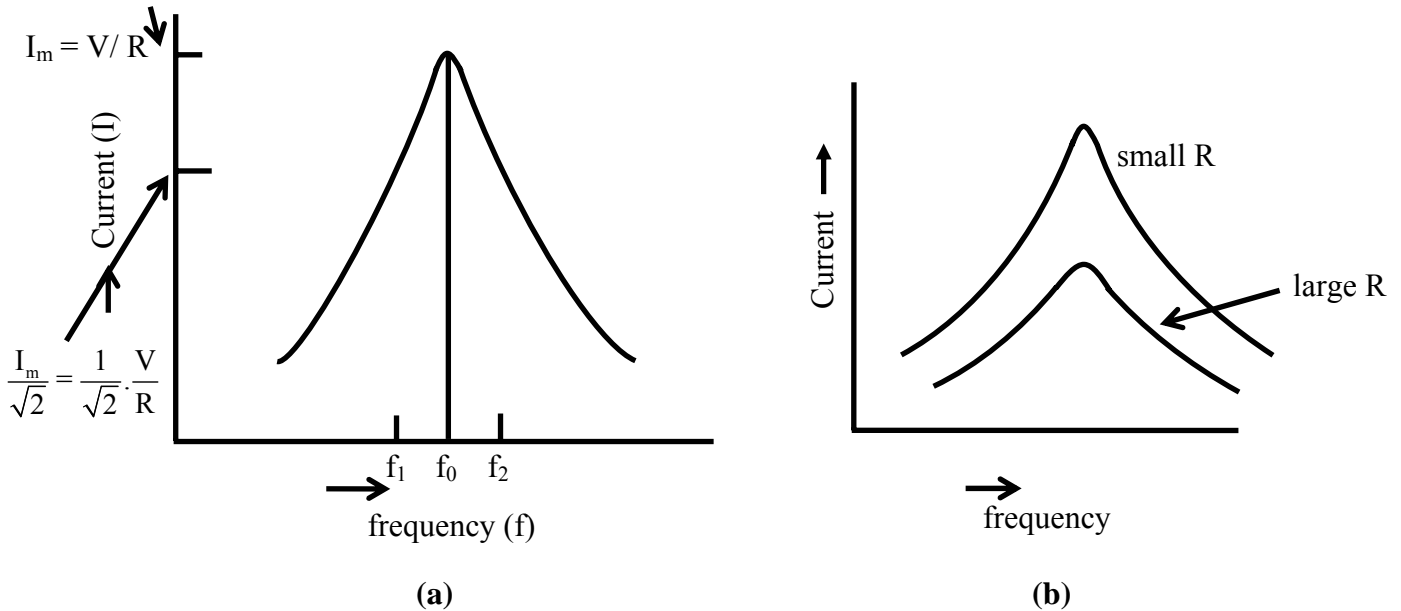
$$Q = \frac{\omega_o L}{R} = \frac{2\pi f_o L}{R} = \frac{1}{R} \sqrt{\frac{L}{C}}$$



**Fig. 17.1 (b) Phasor Diagram**

It is termed as Quality (Q) factor of the coil.

The impedance of the circuit with the constant values of inductance  $L$ , and capacitance  $C$  is minimum at resonant frequency ( $f_o$ ), and increases as the frequency is changed, i.e. increased or decreased, from the above frequency. The current is maximum at  $f = f_o$ , and decreases as frequency is changed ( $f > f_o$ , or  $f < f_o$ ), i.e.  $f \neq f_o$ . The variation of current in the circuit having a known value of capacitance with a variable frequency supply is shown in Fig. 17.2.



**Fig. 17.2 Variation of current under variable frequency supply**

The maximum value of the current is  $(V/R)$ . If the magnitude of the current is reduced to  $(1/\sqrt{2})$  of its maximum value, the power consumed in  $R$  will be half of that with the maximum current, as power is  $I^2 R$ . So, these points are termed as half power

points. If the two frequencies are taken as  $f_1$  and  $f_2$ , where  $f_1 = f_0 - \Delta f / 2$  and  $f_2 = f_0 + \Delta f / 2$ , the band width being given by  $\Delta f = f_2 - f_1$ .

The magnitude of the impedance with the two frequencies is

$$Z = \left[ R^2 + \left( 2\pi(f_0 \pm \Delta f / 2)L - \frac{1}{2\pi(f_0 \pm \Delta f / 2)C} \right)^2 \right]^{\frac{1}{2}}$$

As  $(2\pi f_0 L = 1/2\pi f_0 C)$  and the ratio  $(\Delta f / 2 f_0)$  is small, the magnitude of the reactance of the circuit at these frequencies is  $X = X_{L0}(\Delta f / f_0)$ . As the current is  $(1/\sqrt{2})$  of its maximum value, the magnitude of the impedance is  $(\sqrt{2})$  of its minimum value (R) at resonant frequency.

$$\text{So, } Z = \sqrt{2} \cdot R = \left[ R^2 + (X_{L0}(\Delta f / f_0))^2 \right]^{\frac{1}{2}}$$

From the above, it can be obtained that  $(\Delta f / f_0)X_{L0} = R$

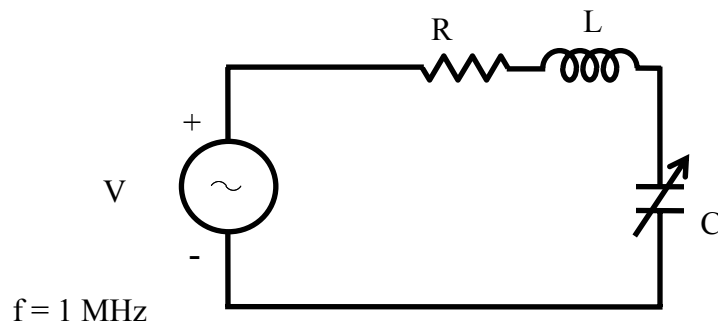
$$\text{or } \Delta f = f_2 - f_1 = \frac{R f_0}{X_{L0}} = \frac{R f_0}{2\pi f_0 L} = \frac{R}{2\pi L}$$

The band width is given by  $\Delta f = f_2 - f_1 = R/(2\pi L)$

It can be observed that, to improve the quality factor (Q) of a coil, it must be designed to have its resistance, R as low as possible. This also results in reduction of band width and losses (for same value of current). But if the resistance, R cannot be decreased, then Q will decrease, and also both band width and losses will increase.

### Example 17.1

A constant voltage of frequency, 1 MHz is applied to a lossy inductor (r in series with L), in series with a variable capacitor, C (Fig. 17.3). The current drawn is maximum, when  $C = 400$  pF; while current is reduced to  $(1/\sqrt{2})$  of the above value, when  $C = 450$  pF. Find the values of r and L. Calculate also the quality factor of the coil, and the bandwidth.



**Fig. 17.3 Circuit diagram**

### Solution

$$f = 1 \text{ MHz} = 10^6 \text{ Hz}$$

$$\omega = 2\pi f$$

$$C = 400 \text{ pF} = 400 \cdot 10^{-12} \text{ F}$$

$$I_{\max} = V / r \quad \text{as } X_L = X_C \quad X_c = \frac{1}{2\pi f C} = \frac{1}{2\pi \cdot 10^6 \times 400 \cdot 10^{-12}} = 398 \, \Omega$$

$$X_L = X_C = 2\pi f L = 398 \, \Omega \quad L = \frac{398.0}{2\pi \cdot 10^6} = 63.34 \, \mu H$$

$$C_1 = 450 \, pF \quad X_{C1} = \frac{1}{2\pi \cdot 10^6 \times 450 \cdot 10^{-12}} = 353.7 \, \Omega$$

$$Z \angle \phi = r + j(X_L - X_{C1}) = r + j(398.0 - 353.7) = (r + j44.3) \, \Omega$$

$$I = \frac{I_{\max}}{\sqrt{2}} = \frac{V}{\sqrt{2} \cdot r} = \frac{V}{Z} = \frac{V}{\sqrt{r^2 + (44.3)^2}}$$

$$\text{From above, } \sqrt{2} \cdot r = \sqrt{r^2 + (44.3)^2} \quad \text{or } 2r^2 = r^2 + (44.3)^2 \\ \text{or } r = 44.3 \, \Omega$$

$$\text{The quality factor of the coil is } Q = \frac{X_L}{r} = \frac{398.0}{44.3} = 8.984$$

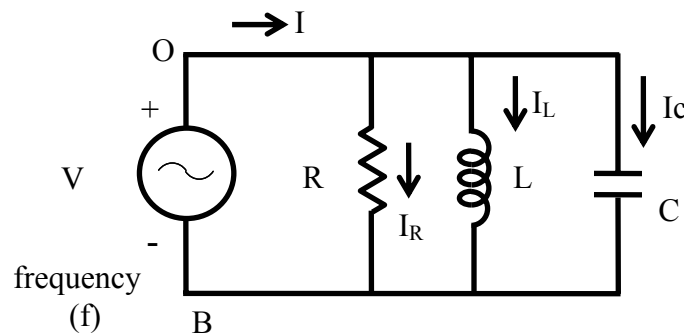
The band width is

$$\Delta f = f_2 - f_1 = \frac{r}{2\pi L} = \frac{44.3}{2\pi \times 63.34 \cdot 10^{-6}} = \frac{44.3}{398 \cdot 10^{-6}} = 0.1113 \cdot 10^6 = 0.1113 \, MHz \\ = 111.3 \cdot 10^3 = 111.3 \, kHz$$

## Parallel circuit

The circuit, with resistance R, inductance L, and a capacitor, C in parallel (Fig. 17.4a) is connected to a single phase variable frequency (f) supply.

The total admittance of the circuit is



**Fig. 17.4 (a) Circuit diagram.**

$$Y \angle \phi = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

where,

$$Y = \sqrt{\left[ \frac{1}{R^2} + \left( \omega C - \frac{1}{\omega L} \right)^2 \right]} ; \quad \phi = \tan^{-1} \left[ R \left( \omega C - \frac{1}{\omega L} \right) \right] ; \quad \omega = 2\pi f$$

The impedance is  $Z \angle -\phi = 1/Y \angle \phi$

The current is

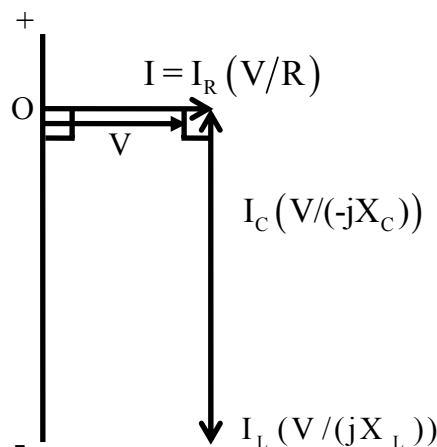
$$I \angle \phi = V \angle 0^\circ \cdot Y \angle \phi = (V \cdot Y) \angle \phi = V \angle 0^\circ / Z \angle -\phi = (V/Z) \angle \phi$$

$$\text{where, } I = V \sqrt{\left[ \frac{1}{R^2} + \left( \omega C - \frac{1}{\omega L} \right)^2 \right]}$$

The current in the circuit is minimum, if  $\omega C = \frac{1}{\omega L}$

The frequency under the above condition is

$$f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$



**Fig. 17.4 (b) Phasor Diagram**

This condition under which the magnitude of the total (supply) current is minimum, or the magnitude of the admittance is minimum (which means that the impedance is maximum), is called resonance. It may be noted that, for parallel circuit, the current or admittance is minimum (the impedance being maximum), while for series circuit, the current is maximum (the impedance being minimum). The frequency under this condition with the constant values of inductance  $L$ , and capacitance  $C$ , is called resonant frequency. If the capacitance is variable, and the frequency,  $f$  is kept constant, the value of the capacitance needed to produce this condition is

$$C = \frac{1}{\omega^2 L} = \frac{1}{(2\pi f)^2 L}$$

The magnitude of the impedance under the above condition is ( $|Z| = R$ ), while the magnitude of the admittance is ( $|Y| = G = (1/R)$ ). The reactive part of the admittance is



$B = 0$ , as the susceptance (inductive)  $B_L = (1/\omega L)$  is equal to the susceptance (capacitive)  $B_C = \omega C$ . The phase angle is  $\phi = 0^\circ$ , and the power factor is unity ( $\cos \phi = 1$ ). The total (supply) current is phase with the input voltage. So, the magnitude of the total current ( $|V/R|$ ) in the circuit is only limited by resistance  $R$ . The phasor diagram is shown in Fig. 17.4b.

The magnitude of the current in the inductance,  $L$  / capacitance,  $C$  (both are equal, as the reactance are equal), is  $V(1/\omega_o L) = V \cdot \omega_o C$ . This may be termed as the circulating current in the circuit with only inductance and capacitance, the magnitude of which is

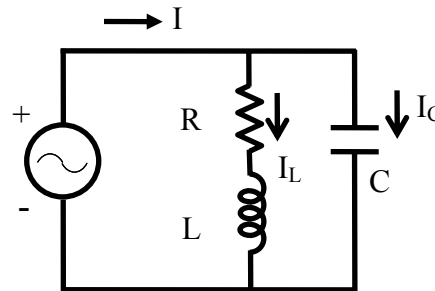
$$|I_L| = |I_C| = V \sqrt{\frac{C}{L}}$$

substituting the value of  $\omega_o = 2\pi f_o$ . This circulating current is smaller in magnitude than the input current or the current in the resistance as  $\omega_o C = (1/\omega_o L) > R$ .

The input current increases as the frequency is changed, i.e. increased or decreased from the resonant frequency ( $f > f_o$ , or  $f < f_o$ ), i.e.  $f \neq f_o$ .

In the two cases of series and parallel circuits described earlier, all components, including the inductance, are assumed to be ideal, which means that the inductance is lossless, having no resistance. But, in actual case, specially with an iron-cored choke coil, normally a resistance  $r$  is assumed to be in series with the inductance  $L$ , to take care of the winding resistance and also the iron loss in the core. In an air-cored coil, the winding resistance may be small and no loss occurs in the air core.

An iron-cored choke coil is connected in parallel to capacitance, and the combination is fed to an ac supply (Fig. 17.5a).



**Fig. 17.5 (a) Circuit diagram.**

The total admittance of the circuit is

$$Y = Y_1 + Y_2 = \frac{1}{r + j\omega L} + j\omega C = \frac{r - j\omega L}{r^2 + \omega^2 L^2} + j\omega C$$

If the magnitude of the admittance is to be minimum, then

$$\omega C = \frac{\omega L}{r^2 + \omega^2 L^2} \text{ or } C = \frac{L}{r^2 + \omega^2 L^2}.$$

The frequency is

$$f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{L}{C} - r^2}$$

This is the resonant frequency. The total admittance is  $Y \angle 0^\circ = \frac{r}{r^2 + \omega^2 L^2}$

The total impedance is  $Z \angle 0^\circ = \frac{r^2 + \omega^2 L^2}{r}$

The total (input) current is

$$I \angle 0^\circ = \frac{V \angle 0^\circ}{Z \angle 0^\circ} = V \angle 0^\circ \cdot Y \angle 0^\circ = \left( \frac{V}{Z} \right) \angle 0^\circ = (V \cdot Y) \angle 0^\circ = \frac{V \cdot r}{r^2 + \omega^2 L^2}$$

This current is at unity power factor with  $\phi = 0^\circ$ . The total current can be written as

$$I \angle 0^\circ = I + j0 = I_L \angle -\phi_L + jI_C = I_L \cos \phi_L + j(I_L \sin \phi_L - I_C)$$

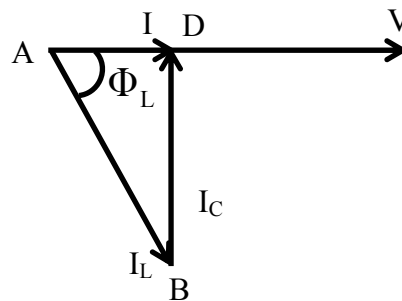
So, the condition is  $I_C = I_L \sin \phi_L$

$$\text{where } I_C = \frac{V}{X_C} = V \cdot \omega C ; \quad I_L = \frac{V}{\sqrt{r^2 + \omega^2 L^2}} ; \quad \sin \phi_L = \frac{\omega L}{\sqrt{r^2 + \omega^2 L^2}}$$

From the above, the condition, as given earlier, can be obtained.

The total current is  $I = I_L \cos \phi_L$

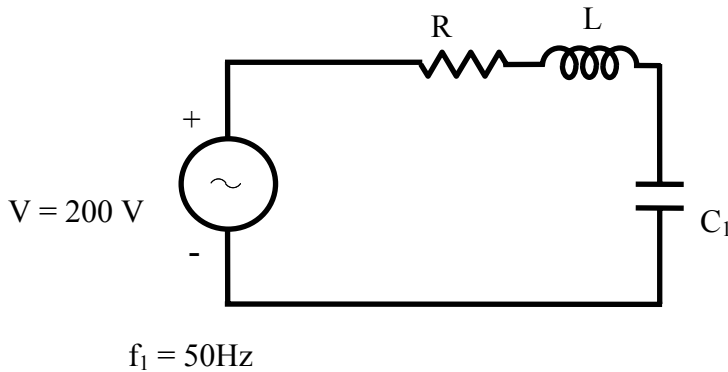
The value, as given here, can be easily obtained. The phasor diagram is shown in Fig. 17.5b. It may also be noted that the magnitude of the total current is minimum, while the magnitude of the impedance is maximum.



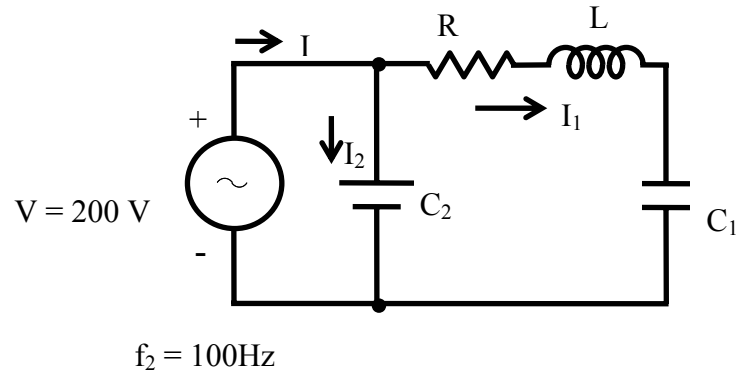
**Fig. 17.5 (b) Phasor Diagram**

### Example 17.2

A coil, having a resistance of  $15 \, \Omega$  and an inductance of  $0.75 \, \text{H}$ , is connected in series with a capacitor (Fig. 17.6a). The circuit draws maximum current, when a voltage of  $200 \, \text{V}$  at  $50 \, \text{Hz}$  is applied. A second capacitor is then connected in parallel to the circuit (Fig. 17.6b). What should be its value, such that the combination acts like a non-inductive resistance, with the same voltage ( $200 \, \text{V}$ ) at  $100 \, \text{Hz}$ ? Calculate also the current drawn by the two circuits.



**Fig. 17.6 (a) Circuit diagram**



**Fig. 17.6 (b) Circuit diagram**

### Solution

$$f_1 = 50 \text{ Hz} \quad V = 200 \text{ V} \quad R = 15 \, \Omega \quad L = 0.75 \text{ H}$$

From the condition of resonance at 50 Hz in the series circuit,

$$X_{L1} = \omega_1 L = 2\pi f_1 L = X_{C1} = \frac{1}{\omega_1 C_1} = \frac{1}{2\pi f_1 C_1}$$

$$\text{So, } C_1 = \frac{1}{(2\pi f_1)^2 L} = \frac{1}{(2\pi \cdot 50)^2 \times 0.75} = 13.5 \cdot 10^{-6} = 13.5 \, \mu\text{F}$$

The maximum current drawn from the supply is,  $I_{\max} = V / R = 200 / 15 = 13.33 \text{ A}$

$$f_2 = 100 \text{ Hz} \quad \omega_2 = 2\pi f_2 = 2\pi \cdot 100 = 628.3 \text{ rad/s}$$

$$X_{L2} = 2\pi f_2 L = 2\pi \cdot 100 \cdot 0.75 = 471.24 \, \Omega$$

$$X_{C2} = \frac{1}{2\pi f_2 C_1} = \frac{1}{2\pi \cdot 100 \cdot 13.5 \cdot 10^{-6}} = 117.8 \, \Omega$$

$$Z_1 \angle \phi_1 = R + j(X_{L2} - X_{C2}) = 15 + j(471.24 - 117.8) = 15 + j353.44 \\ = 353.75 \angle 87.57^\circ \, \Omega$$

$$Y_1 \angle -\phi_1 = \frac{1}{Z_1 \angle \phi_1} = \frac{1}{15 + j353.44} = \frac{1}{353.75 \angle 87.57^\circ} = 2.827 \cdot 10^{-3} \angle -87.57^\circ$$

$$= (0.12 - j2.824) \cdot 10^{-3} \, \Omega^{-1}$$

$$Y_2 = 1/Z_2 = j(\omega_2 C_2)$$

As the combination is resistive in nature, the total admittance is

$$Y \angle 0^\circ = Y + j0 = Y_1 + Y_2 = (0.12 - j2.824) \cdot 10^{-3} + j\omega_2 C_2$$

From the above expression,  $\omega_2 C_2 = 628.3 \cdot C_2 = 2.824 \cdot 10^{-3}$

$$\text{or, } C_2 = \frac{2.824 \cdot 10^{-3}}{628.3} = 4.5 \cdot 10^{-6} = 4.5 \, \mu\text{F}$$

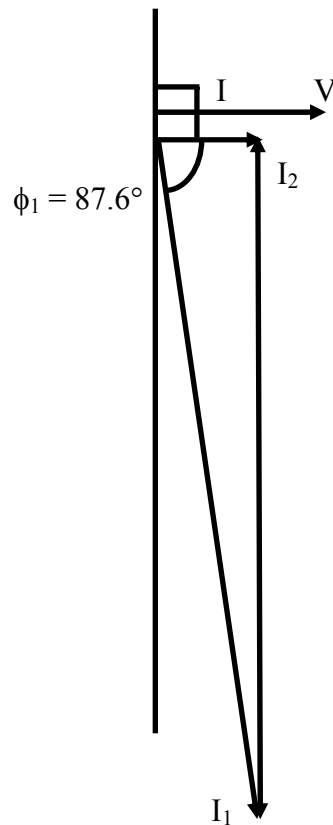
The total admittance is  $Y = 0.12 \cdot 10^{-3} \, \Omega^{-1}$

The total impedance is  $Z = 1/Y = 1/(0.12 \cdot 10^{-3}) = 8.33 \cdot 10^3 \, \Omega = 8.33 \, \text{k}\Omega$

The total current drawn from the supply is

$$I = V \cdot Y = V / Z = 200 \times 0.12 \cdot 10^{-3} = 0.024 \text{ A} = 24 \cdot 10^{-3} = 24 \text{ mA}$$

The phasor diagram for the circuit (Fig. 17.6b) is shown in Fig. 17.6c.

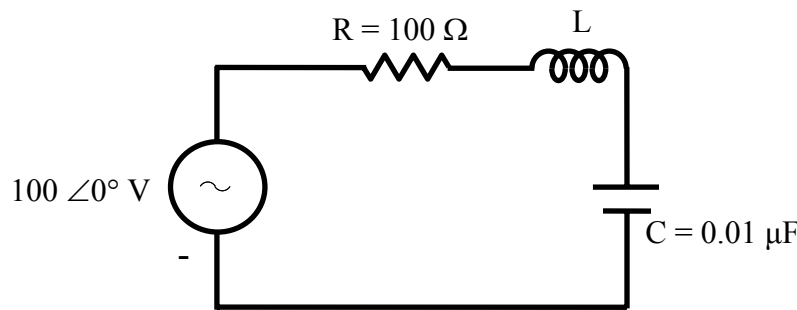


**Fig. 17.6 (c) Phasor diagram**

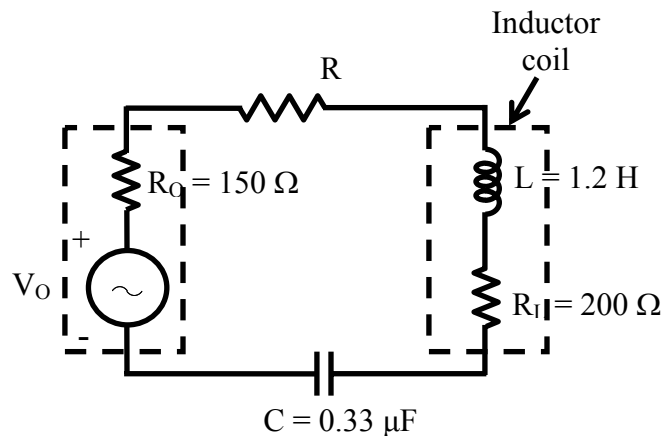
The condition for resonance in both series and parallel circuits fed from single phase ac supply is described. It is shown that the current drawn from the supply is at unity power factor (upf) in both cases. The value of the capacitor needed for resonant condition with a constant frequency supply, and the resonant frequency with constant value of capacitance, have been derived. Also taken up is the case of a lossy inductance coil in parallel with a capacitor under variable frequency supply, where the total current will be at upf. The quality factor of the coil and the bandwidth of the series circuit with known value of capacitance have been determined. This is the final lesson in this module of single phase ac circuits. In the next module, the circuits fed from three phase ac supply will be described.

## Problems

- 17.1 A coil having a resistance of  $20\ \Omega$  and inductance of  $20\ \text{mH}$ , in series with a capacitor is fed from a constant voltage variable frequency supply. The maximum current is  $10\ \text{A}$  at  $100\ \text{Hz}$ . Find the two cut-off frequencies, when the current is  $0.71\ \text{A}$ .
- 17.2 With the ac voltage source in the circuit shown in Fig. 17.7 operating a frequency of  $f$ , it was found that  $I = 1.0 \angle 0^\circ\ \text{A}$ . When the source frequency was doubled ( $2f$ ), the current became  $I = 0.707 \angle -45^\circ\ \text{A}$ . Find:
- The frequency  $f$ , and
  - The inductance  $L$ , and also the reactances,  $X_L$  and  $X_C$  at  $2f$
- 17.3 For the circuit shown in Fig. 17.8,
- Find the resonant frequency  $f_0$ , if  $R = 250\ \Omega$ , and also calculate  $Q_0$  (quality factor), BW (band width) in Hz, and lower and upper cut-off frequencies ( $f_1$  and  $f_2$ ) of the circuit.
  - Suppose it was desired to increase the selectivity, so that BW was  $65\ \text{Hz}$ . What value of  $R$  would accomplish this?

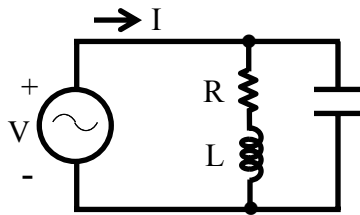


**Fig. 17.7**

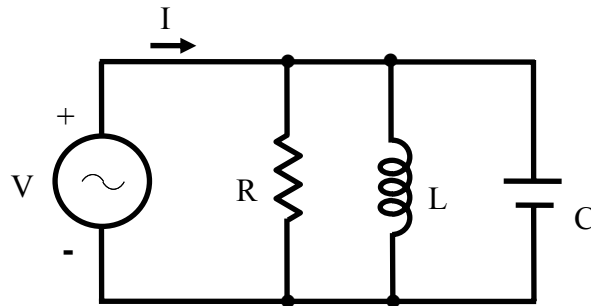


**Fig. 17.8**

- 17.4 (a) For the circuit shown in Fig. 17.9, show that the circulating current is given by  $V \cdot \sqrt{C/L}$ , if  $R$  is small and  $V$  is the input voltage.  
 (b) Find the total current at  
 (i) resonant frequency,  $f_0$ , and  
 (ii) at a frequency,  $f_1 = 0.9 f_0$ .
- 17.5 The circuit components of a parallel circuit shown in Fig. 17.10 are  $R = 60 \text{ k}\Omega$ ,  $L = 5 \text{ mH}$ , and  $C = 50 \text{ pF}$ . Find  
 a) the resonant frequency,  $f_0$ ,  
 b) the quality factor,  $Q_0$ , and  
 c) the bandwidth.



**Fig. 17.9**



**Fig. 17.10**

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